PSEUDO-HERMITICITY OF AN EXACTLY SOLVABLE TWO-DIMENSIONAL MODEL

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We study a two-dimensional exactly solvable non-Hermitian $PT-$non-symmetric quantum model with real spectrum, which is not amenable to separation of variables, by supersymmetrical methods. Here we focus attention on the property of pseudo-Hermiticity, biorthogonal expansion and pseudo-metric operator. To our knowledge this is the first time that pseudo-Hermiticity is realized explicitly for a nontrivial two-dimensional case. It is shown that the Hamiltonian of the model is not diagonalizable.

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1 Introduction.

Supersymmetrical techniques has been successfully applied to two-dimensional Quantum Mechanics (see [1] and the review paper [2]). For the Hermitian case several **real** two-dimensional models - Morse potential [3], Pöschl-Teller potential [4] and some others [5] - were studied by means of two different SUSY methods: SUSY-separation of variables and shape invariance. Some partial solutions of the spectral problems were obtained by this approach.

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During the last decade intensive study of Schrödinger equation with complex potentials, but with real spectrum, was performed by different methods. The pioneer papers [6] initiated investigation of $PT-$symmetric systems (see also the review papers [7]), and afterwards more general class of pseudo-Hermitian models was considered [8].

One key tool for the **complexification** of two-dimensional models with real spectra is given by the intertwining relations between partner Hamiltonians with supercharges of second order in derivatives. A particular class of models - complex **singular** two-dimensional Morse - has been found [9] to satisfy SUSY-pseudo-Hermiticity, i.e.

$$H^{\dagger}Q^{+} = Q^{+}H, \tag{1}$$

where the complex supercharges intervene in the HSUSY deformation of the standard algebra of SUSY QM. Only partial knowledge of the spectrum and wave functions for this model was obtained.

More recently another complexification of the **real** singular Morse model was considered [10], which from now on we will call **regularized** complex Morse system. This Hamiltonian being also involved in second order SUSY intertwining **does not** satisfy (1), but we point out that it fulfills standard pseudo-Hermiticity as defined in [8]:

$$H^{\dagger} = \eta H \eta^{-1} \tag{2}$$

with some invertible positive-definite operator η. In particular, since in this model the complexification arises from a complex coordinate shift, which also provides a regularization, pseudo-Hermiticity is rather straightforward [11]. Due to the fact that this model turns out to be solvable, we can now discuss explicitly the biorthogonal expansion based on the eigenfunctions and their complex conjugated.

The structure of the paper is the following. The main results concerning the partially solvable model with real two-dimensional Morse potential are reproduced in Section 2. In Section 3 the exactly solvable two-dimensional regularized complex Morse model is analysed. This model is not $PT-$symmetric, but its energy eigenvalues are real. After the description of its

spectrum and corresponding eigenfunctions, the biorthogonal basis based on these eigenfunctions and their complex conjugated is studied. The explicit construction indicates that the Hamiltonian is actually not diagonalizable (concerning one-dimensional non-diagonalizable Hamiltonians see, for example, [12] and references therein), and biorthogonal basis has to be completed in order to provide the resolution of identity. Finally, according to the well known prescriptions, the pseudo-metric operator η and the corresponding positively definite (pseudo)inner product [8], [13] space are displayed explicitly.

2 The partially solvable real two-dimensional Morse potential.

The pseudo-Hermitian model which we want to study in the next Section originates from the following Hermitian Hamiltonians:

$$\tilde{H}(\vec{x}) = -\Delta + \frac{\alpha^2 a(2a-1)}{\sinh^2\left(\frac{\alpha x_-}{2}\right)} + A\Big[e^{-2\alpha x_1} - 2e^{-\alpha x_1} + e^{-2\alpha x_2} - 2e^{-\alpha x_2}\Big] + 4a^2\alpha^2, \tag{3}$$

$$H(\vec{x}) = -\Delta + \frac{\alpha^2 a(2a+1)}{\sinh^2\left(\frac{\alpha x_-}{2}\right)} + A\Big[e^{-2\alpha x_1} - 2e^{-\alpha x_1} + e^{-2\alpha x_2} - 2e^{-\alpha x_2}\Big] + 4a^2\alpha^2, \tag{4}$$

where the parameters $a, A, \alpha > 0$ are arbitrary real numbers, and $x_\pm \equiv x_1 \pm x_2$.

For the particular choice of parameter $a = -1/2$, SUSY intertwining relations

$$Q^+H = \tilde{H}Q^+; \quad HQ^- = Q^-\tilde{H} \tag{5}$$

with the supercharges:

$$\begin{aligned} Q^\pm &= 4\partial_+\partial_- \mp 2\alpha\partial_- \mp 2\alpha\coth\frac{\alpha x_-}{2}\partial_+ + \alpha^2\coth\frac{\alpha x_-}{2} - \\ &- A\Big[e^{-2\alpha x_1} - 2e^{-\alpha x_1} - e^{-2\alpha x_2} + 2e^{-\alpha x_2}\Big]; \quad \partial_\pm = \frac{\partial}{\partial x_\pm} \end{aligned} \tag{6}$$

link the Hermitian Hamiltonian $\tilde{H}(\vec{x})$ from (3) to a partner $H(\vec{x})$ of (4) which does not contain the first term in the r.h.s of (4) and therefore allows for separation of variables [10].

The latter Hamiltonian $H(\vec{x})$ is straightforwardly solvable with energies expressed in terms of two integer positive numbers:

$$E_{n,m} = \epsilon_n + \epsilon_m + \alpha^2; \quad \epsilon_k \equiv -A[1 - \frac{\alpha}{\sqrt{A}}(k+1/2)]^2; \quad k,n,m = 0,1,\ldots \tag{7}$$

All levels of H with $n \neq m$ are two-fold degenerated and the corresponding wave functions can be chosen as symmetric and antisymmetric combinations:

$$\Psi^S_{n,m} = \Psi_{n,m} + \Psi_{m,n}; \qquad \Psi^A_{n,m} = \Psi_{n,m} - \Psi_{m,n}, \tag{8}$$

where the functions $\Psi_{n,m}$ were defined as:

$$\Psi_{n,m} = \eta_n(x_1)\eta_m(x_2). \tag{9}$$

In turn, η_k are the standard solutions of the one-dimensional Morse problem and can be written in terms of confluent hypergeometric functions:

$$\Big(-\partial^2 + A\Big(\exp(-2\alpha x) - 2\exp(-\alpha x)\Big)\Big)\eta_n(x) = \epsilon_n\eta_n(x); \tag{10}$$

$$\eta_n = \exp(-\frac{\xi}{2})(\xi)^{s_n}\Phi(-n, 2s_n+1; \xi); \quad \xi \equiv \frac{2\sqrt{A}}{\alpha}\exp(-\alpha x); \tag{11}$$

$$s_n = \frac{\sqrt{A}}{\alpha} - n - 1/2 > 0; \quad \epsilon_n = -\alpha^2 s_n^2; \quad n = 0,1,\ldots \tag{12}$$

The wave functions of $\tilde{H}(\vec{x})$ with the same energies (7) are obtained from (8) acting by supercharge Q^+ from (6):

$$\tilde{\Psi}^{A(S)}_{n,m} = Q^+\Psi^{S(A)}_{n,m}. \tag{13}$$

The operator Q^+ has singular coefficient functions, and it is antisymmetric under $x_1 \Leftrightarrow x_2$.

The two-fold degeneracy of levels (7) of H under $n \leftrightarrow m$ is not reproduced, in general, in the spectrum of $\tilde{H}$. While the singularities of Q^+ at $x_- = 0$ can be compensated by $\Psi^A_{n,m}$ for $\tilde{\Psi}^S_{n,m}(\vec{x})$, the wave functions $\tilde{\Psi}^A_{n,m}$ may be nonnormalizable. Up to now the hypergeometric functions in expressions for the wave functions (9) **did not allow** to perform a comprehensive analysis of the normalizability of **all** wave functions, i.e. to prove the exact solvability of the model (see details and some examples in [10]).

3 The regularized complex Morse model.

In order to avoid the singularities at $x_- \to 0$, which hinder the solvability of the model, it is useful [10] to perform a suitable complex coordinate shift

$$\vec{x} \to \vec{x} + i\vec{\delta}; \quad \vec{\delta} = (\delta, 0) \tag{14}$$

with δ small enough (such that $\alpha\delta \in (0, \pi/2)$) in order to remove the singularities from the real (x_1, x_2) plane, preserving the normalizability of the functions $\eta_n(x_1)$ from (11) at $x_1 \to -\infty$. After this complex shift the Hamiltonian has obviously a real spectrum, but the analysis of normalizability of wave functions is now essentially simplified by the absence of singularities in the supercharges. Complexification of both operators $Q^{\pm}$ is achieved by the same shift (14) in definitions (6). Therefore, their mutual Hermitian conjugacy is replaced now by

$$Q^- = ((Q^+)^\dagger)^\star. \tag{15}$$

The spectrum of the complexified Hamiltonian $H(\vec{x}+i\vec{\delta})$, which is still amenable to separation of variables, coincides with (7), and **all** eigenfunctions $\Psi_{n,m}$ are obtained from (9) by the same imaginary shift of $\vec{x}$.

Similarly to the Hermitian case, the intertwining relations (5) lead to the eigenfunctions $\tilde{\Psi}_{n,m}(\vec{x} + i\vec{\delta})$ of the non-separable non-Hermitian Hamiltonian $\tilde{H}(\vec{x} + i\vec{\delta})$:

$$\tilde{\Psi}^{A(S)}_{n,m}(\vec{x} + i\vec{\delta}) = Q^+(\vec{x} + i\vec{\delta})\Psi^{S(A)}_{n,m}(\vec{x} + i\vec{\delta}), \tag{16}$$

but now, due to the absence of singularity of Q^+ at $x_- \to 0$, these wave functions are normalizable. The corresponding eigenvalues (see (7)) are two-fold degenerate: one can choose symmetric or antisymmetric combinations of $\Psi_{n,m}$. The only exclusions are the levels $E_{n,n\pm1}$, which are not degenerate, because antisymmetric functions $\Psi^A_{n,n\pm1}$, being [10] the linear combinations of zero modes of Q^+, are annihilated by Q^+.

It is known [1] that both Hamiltonians H and $\tilde{H}$ obey the dynamical symmetry properties. The fourth order operators $R = Q^-Q^+$ and $\tilde{R} = Q^+Q^-$ commute with H and $\tilde{H}$, respectively,

while they do not mix the degenerate wave functions. For the case of complex potentials these operators are not Hermitian because of the relation (15).

In next Section we will need the eigenvalues $r_{n,m}$ of R for eigenfunctions $\Psi^{S(A)}_{n,m}(\vec{x}+i\vec{\delta})$. They can be calculated in terms of "one-dimensional energies" ϵ_n, ϵ_m of (7). Indeed, separation of variables in operator H with $a=-1/2$ gives:

$$\begin{aligned} H(\vec{x}+i\vec{\delta}) &= h_1(x_1+i\delta)+h_2(x_2)+\alpha^2; && (17)\\ h_1 &= -\partial_1^2 - f_1 = -\partial_1^2 + A\Big(e^{-2\alpha(x_1+i\delta)} - 2e^{-\alpha(x_1+i\delta)}\Big); && (18)\\ h_2 &= -\partial_2^2 + f_2 = -\partial_2^2 + A\Big(e^{-2\alpha x_2} - 2e^{-\alpha x_2}\Big); && (19) \end{aligned}$$

The explicit form (6) of the supercharges $Q^{\pm}$ leads to the following expression:

$$R = Q^-Q^+ = \Big(h_2 - h_1 + \frac{1}{4}C_+C_- - C_+\partial_- - C_-\partial_+\Big)\Big(h_2 - h_1 + \frac{1}{4}C_+C_- + C_+\partial_- + C_-\partial_+\Big), \quad (20)$$

which for $a=-\frac{1}{2}$ can be transformed by straightforward calculations to:

$$R = (h_1-h_2)^2 + 2\alpha^2(h_1+h_2) + \alpha^4. \quad (21)$$

It means that the eigenvalues $r_{n,m}$ are expressed as:

$$r_{n,m} = (\epsilon_n - \epsilon_m)^2 + 2\alpha^2(\epsilon_n+\epsilon_m) + \alpha^4 = \alpha^4\Big((m-n)^2 - 1\Big)\Big((s_m+s_n)^2 - 1\Big), \quad (22)$$

where the positive parameters s_n were defined in (12). One can notice that for some integer n, m eigenvalues $r_{n,m}$ are not positive (operator R is not Hermitian): $r_{n,n} = \alpha^4(1-4s_n^2) < 0$ for all n (excluding $n = [\frac{\sqrt{A}}{\alpha} - \frac{1}{2}]$), and $r_{n,n\pm 1} = 0$ for all values of n.

In general, besides eigenfunctions of the form (16) some *additional* normalizable eigenstates of $\tilde{H}$ could exist, if they would be annihilated by Q^-, or if they would be transformed by Q^- into nonnormalizable functions. The second option is excluded due to nonsingular form of supercharges. The analysis of zero modes of Q^- is performed analogously to investigation in [3] (Subsections 4.3 - 4.5) but up to some appropriate changes in that paper: $Q^+ \to Q^-$; $h \to \tilde{h}$ etc[d]. The required set of $\tilde{\Psi}_n$ - linear combinations of N zero modes

[d] In particular, it means that one has to use in these calculations $a = 1/2$.

$\Omega_l;\ l = 0, 1, \ldots, N$ of Q^- - is constructed by means of SUSY-separation of variables [3] and the similarity transformation with function $\xi_1\xi_2(\xi_2 - \xi_1)^{-1}$:

$$\tilde{\Psi}_n = \sum_{k=0}^{N} b_{nl}\Omega_l, \tag{23}$$

where b_{nl} are matrix elements of $\hat{B}$, which satisfy the matrix equation:

$$\hat{E}\hat{B} = \hat{B}\hat{C}. \tag{24}$$

In this equation $\hat{E}$ is diagonal matrix with elements

$$E_n = c_{nn} = -2\alpha^2 s_n(1 + s_n); \quad n = 0, 1, 2, \ldots, N, \tag{25}$$

and $\hat{C}$ is the triangular matrix [3] with elements c_{nk}, such that

$$\tilde{H}\Omega_n = \sum_{k=0}^{N} c_{nk}\Omega_k. \tag{26}$$

The direct algorithm for calculation of b_{nl} in terms of known c_{nk} was also given in [3].

One can notice that the eigenvalues E_n from (25) for the values $n = 1, 2, \ldots, N$ coincide with the eigenvalues $E_{n-1,n}$ of $\tilde{\Psi}_{n-1,n} = Q^+\Psi^S_{n-1,n}$ from (16), which were found by using intertwining relations. It is necessary now to compare the corresponding eigenfunctions $\tilde{\Psi}_n$ and $\tilde{\Psi}_{n-1,n}$.

Since the eigenvalues $r_{n-1,n}$ of $R = Q^-Q^+$ vanish for all $n = 1, 2, \ldots, N$, the eigenfunctions $\tilde{\Psi}_{n-1,n}$ of $\tilde{H}$ are simultaneously the zero modes of Q^-, and therefore must be linear combinations of Ω_k with some unknown coefficients a_{nk}:

$$Q^+\Psi^S_{n-1,n} = \sum_{k=0}^{N} a_{nk}\Omega_k; \quad n = 1, 2, \ldots, N. \tag{27}$$

Acting with the $\tilde{H}$ onto both sides of this relation and subsequently equating coefficients in front of Ω_l gives:

$$E_{n-1,n}a_{nl} = \sum_{k=0}^{N} a_{nk}c_{kl}; \quad n, l = 1, 2, \ldots, N. \tag{28}$$

In matrix form this equation coincides with (24) up to replacing b_{nk} by a_{nk}, therefore these matrix elements also coincide up to a common constant factor. This analysis shows that functions $Q^+\Psi^S_{n-1,n}$ coincide with $\tilde{\Psi}_n$ for $n = 1, 2, \ldots N$, and the eigenvalues $E_{n-1,n} = E_n$ still are not degenerate.

There is *only one* additional eigenstate in the spectrum of $\tilde{H}$ not obtained from intertwining relations. It corresponds to $n = 0$, i.e. $E_0 = -2\alpha^2 s_0(1 + s_0)$. Its wave function - the lowest zero mode of Q^- - reads:

$$\tilde{\Psi}_0 = exp(-\frac{\xi_1 + \xi_2}{2})(\xi_1\xi_2)^{s_0+1}(\xi_2 - \xi_1)^{-1}. \tag{29}$$

Thus, the spectrum of the Hamiltonian $\tilde{H}(\vec{x}+i\vec{\delta})$ is known: it includes two-fold degenerate levels $E_{n,m}$ with $m \neq n \pm 1$, non-degenerate levels $E_{n-1,n}$ with $n = 1, 2, \ldots, N$ and one additional level with energy E_0.

4 Biorthogonal basis and pseudo-Hermiticity.

The wave functions $\Psi^{S(A)}_{n,m}(\vec{x} + i\vec{\delta})$ of $H(\vec{x} + i\vec{\delta})$ (with separation of variables) and their complex conjugate functions $(\Psi^{S(A)}_{n,m}(\vec{x} + i\vec{\delta}))^\star$ form the so called biorthogonal basis for the non-Hermitian Hamiltonian H. The corresponding biorthogonality relations

$$\begin{aligned} \langle \Psi^\star_{n,m} \mid \Psi_{n',m'} \rangle &= \int d^2x \Psi_{n,m}(\vec{x} + i\vec{\delta})\Psi_{n',m'}(\vec{x} + i\vec{\delta}) = \\ &= \int dx_1 \phi_n(x_1 + i\delta)\phi_{n'}(x_1 + i\delta) \int dx_2 \phi_n(x_2)\phi_{n'}(x_2) = \delta_{nn'}\delta_{mm'} \end{aligned} \tag{30}$$

can be checked straightforwardly and by comparing the integral along the line $x_1 \in (-\infty + i\delta, +\infty + i\delta)$ with the integral along the real axis (with no singularities of integrand between these lines).

The construction of the bound-states-biorthogonal basis by means of the wave functions $\tilde{\Psi}_{n,m}(\vec{x} + i\vec{\delta})$ and $\tilde{\Psi}_0(\vec{x} + i\vec{\delta})$ of $\tilde{H}(\vec{x} + i\vec{\delta})$ together with $\left(\tilde{\Psi}_{n,m}(\vec{x} + i\vec{\delta})\right)^\star$ and $(\tilde{\Psi}_0(\vec{x} + i\vec{\delta}))^\star$ is much less simple.

Due to the property (15), for the complex model the scalar products analogous to (30) can be written as:

$$\begin{aligned}
&\langle\Big(\tilde{\Psi}_{n,m}(\vec{x}+i\vec{\delta})\Big)^{\star} \mid \tilde{\Psi}_{n',m'}(\vec{x}+i\vec{\delta})\rangle = \\
&= \langle (Q^{+})^{\star}\Big(\Psi_{n,m}(\vec{x}+i\vec{\delta})\Big)^{\star} \mid Q^{+}\Psi_{n',m'}(\vec{x}+i\vec{\delta})\rangle = \\
&= \langle \Psi^{\star}_{n,m}(\vec{x}+i\vec{\delta}) \mid Q^{-}Q^{+}\Psi_{n',m'}(\vec{x}+i\vec{\delta})\rangle = r_{n,m}\delta_{n,n'}\delta_{m,m'}.
\end{aligned} \tag{31}$$

In the last equality we used the fact that wave functions $\Psi_{n,m}$ are the common eigenfunctions both of the Hamiltonian H and of its symmetry operator $R = Q^{-}Q^{+}$ with the real eigenvalues $r_{n,m}$.

For all pairs n, m with $m \neq n \pm 1$ the functions in (31) can be made orthonormal by suitable normalization factors, real or imaginary depending on the sign of $r_{n,m}$. In particular, for $m = n$ one can choose $i|r_{n,n}|^{-1/2}\tilde{\Psi}_{n,n}(\vec{x}+i\vec{\delta})$ and $\Big(i|r_{n,n}|^{-1/2}\tilde{\Psi}_{n,n}(\vec{x}+i\vec{\delta})\Big)^{\star}$ as the elements of biorthogonal basis ($r_{n,n} < 0$).

No analogous simple prescription works for the functions $\tilde{\Psi}_{n,n\pm1}(\vec{x}+i\vec{\delta})$. The zero value of $r_{n,n\pm1}$, i.e. the zero value of the integral $\int\Big(\tilde{\Psi}^{A}_{n,n\pm1}(\vec{x}+i\vec{\delta})\Big)^{2} d^2x$, definitely signals **incompleteness** of the resolution of identity in terms of (nondegenerate) vectors $\tilde{\Psi}_{n,m}, \tilde{\Psi}^{\star}_{n,m}$. In order to give a physical interpretation to the model, one should complete the biorthogonal basis by suitable additional vectors.

Recently the problem of investigation of non-diagonalizable Hamiltonians in one-dimensional Quantum Mechanics with non-Hermitian Hamiltonians was studied in papers [12] (see also the monographs [14]). Up to our knowledge, not much is known about two-dimensional non-diagonalizable Hamiltonians. One can conjecture that the procedure to complete the basis is somehow similar to the one-dimensional case. Then, in addition, one should consider (in the simplest case) the so called (first order) associated functions $\tilde{\Phi}_{n-1,n}(\vec{x}+i\vec{\delta})$, which solve the inhomogeneous equation:

$$(\tilde{H} - E_{n,n\pm1})\tilde{\Phi}_{n,n\pm1} = \tilde{\Psi}_{n,n\pm1}, \tag{32}$$

where the function in r.h.s. is the normalizable eigenfunction of $\tilde{H}$ with eigenvalue $E_{n,n\pm1}$.

Then, due to the second Green's identity (Ostrogradsky-Gauss theorem), the equalities ($\partial/\partial N$ - normal derivative)

$$0 = \int \Big(\tilde{\Psi}_{n,n\pm1}(\vec{x}+i\vec{\delta})\Big)^2 d^2x = \int \Big((\tilde{H} - E_{n,n\pm1})\tilde{\Phi}_{n,n\pm1}\Big)\tilde{\Psi}_{n,n\pm1} d^2x =$$
$$= \int \tilde{\Phi}_{n,n\pm1}\Big((\tilde{H} - E_{n,n\pm1})\tilde{\Psi}_{n,n\pm1}\Big) d^2x - \oint_C (\tilde{\Phi}_{n,n\pm1}\frac{\partial}{\partial N}\tilde{\Psi}_{n,n\pm1}) +$$
$$+ \oint_C (\tilde{\Psi}_{n,n\pm1}\frac{\partial}{\partial N}\tilde{\Phi}_{n,n\pm1}), \qquad (33)$$

demonstrate that the integral over the large contour in the r.h.s. must be zero for arbitrary solution $\tilde{\Phi}_{n,n\pm1}$ of (32), irrespectively of the fact that it is normalizable or not normalizable.

In one dimensional models with discrete spectrum (see [12]) for the normalizable case one can complete the biorthogonal basis with $\tilde{\Phi}_{n,n\pm1}, \tilde{\Phi}^{\star}_{n,n\pm1}$ with corresponding non-diagonal terms in the resolution of identity. Then the Hamiltonian $\tilde{H}$ includes Jordan blocks, and it is called non-diagonalizable.

In the two-dimensional case with discrete spectrum, the general discussion is rather complicated. So, we restrict ourselves to the simplest case $n = 0, m = 1$ in order to provide some analytical insight without ambition to propose general theorems.

In this particular case:

$$(H - E_{0,1})\Phi^S_{0,1} = \Psi^S_{0,1}; \quad E_{0,1} = -2\alpha^2 s_0(s_0 - 1), \qquad (34)$$

where the Hamiltonian with separation of variables is:

$$H = -\alpha^2\Big(\xi_1^2\partial_1^2 + \xi_2^2\partial_2^2 + \xi_1\partial_1 + \xi_2\partial_2 - \frac{1}{4}(\xi_1^2 + \xi_2^2) + (s_0 + \frac{1}{2})(\xi_1 + \xi_2) - 1\Big); \qquad (35)$$
$$\xi_1 = \frac{2\sqrt{A}}{\alpha}\exp{[-\alpha(x_1 + i\delta)]}; \quad \xi_2 = \frac{2\sqrt{A}}{\alpha}\exp{(-\alpha x_2)},$$

and the wave function reads:

$$\Psi^S_{0,1} = \exp{[-\frac{\xi_1 + \xi_2}{2}]}(\xi_1\xi_2)^{s_0}(\frac{1}{\xi_1} + \frac{1}{\xi_2} - \frac{2}{2s_0 - 1}). \qquad (36)$$

It is convenient to look for the particular solution $\Phi_{0,1}$ in the following form:

$$\Phi^S_{0,1} = \exp{[-\frac{\xi_1 + \xi_2}{2}]}(\xi_1\xi_2)^{s_0}(\phi_1(\xi_1) + \phi_2(\xi_2)), \qquad (37)$$

where use has been made of separation of variables in Eq.(34). Correspondingly, one obtains that the function ϕ_1 (and similarly for ϕ_2) fullfils an inhomogeneous ordinary differential equation:

$$-\alpha^2\Big(\xi_1^2\phi_1'' - \xi_1^2\phi_1' + (2s_0-1)\phi_1\Big) = \frac{1}{\xi_1} - \frac{1}{2s_0-1}. \tag{38}$$

The general solution of this equation can be expressed in terms of two linearly independent solutions $y(\xi_1)$ and $z(\xi_1)$ with Wronskian W:

$$\alpha^2\phi_1(\xi_1) = \mu y(\xi_1) + \nu z(\xi_1) + z(\xi_1)\int_0^{\xi_1} d\tau \frac{y(\tau)(\frac{1}{\tau} - \frac{1}{2s_0-1})}{\tau^2 W(\tau)} - y(\xi_1)\int_0^{\xi_1} d\tau \frac{z(\tau)(\frac{1}{\tau} - \frac{1}{2s_0-1})}{\tau^2 W(\tau)}. \tag{39}$$

The analysis of asymptotic behaviour of ϕ_1 and ϕ_2 leads to the conclusion that the function $\Phi_{0,1}^S$ is not normalizable and, in addition, the large contour integral does not vanish. This is expected since the integral of $(\Psi_{0,1})^2$ is different from zero, the biorthogonal basis (30) is complete, and the Hamiltonian H is diagonalizable.

Coming finally to the partner model with the Hamiltonian $\tilde{H}$, we remind that the integral of $(\tilde{\Psi}_{0,1})^2$ is zero. The partner (formal) associated function $\tilde{\Phi}_{0,1} = Q^+\Phi_{0,1}$ turned out to be also nonnormalizable[e], however the large contour integral vanishes as required by (33). Therefore, the problem of completing of the resolution of identity remains open.

The states $\tilde{\Psi}_0$ and $(\tilde{\Psi}_0)^\star$ must also be included in the biorthogonal basis and the resolution of identity. It is easy to show that the state $\tilde{\Psi}_0$ is orthogonal to $(\tilde{\Psi}_{n,m})^\star$:

$$\begin{aligned} &\langle\Big(\tilde{\Psi}_{n,m}(\vec{x}+i\vec{\delta})\Big)^\star \mid \tilde{\Psi}_0(\vec{x}+i\vec{\delta})\rangle = \langle(Q^+)^\star\Big(\Psi_{n,m}(\vec{x}+i\vec{\delta})\Big)^\star \mid \tilde{\Psi}_0(\vec{x}+i\vec{\delta})\rangle = \\ &= \langle\Psi_{n,m}^\star(\vec{x}+i\vec{\delta}) \mid Q^-\tilde{\Psi}_0(\vec{x}+i\vec{\delta})\rangle = 0. \end{aligned} \tag{40}$$

It is difficult to find an analytic expression for the pseudo-norm $\langle(\tilde{\Psi}_0)^\star \mid \tilde{\Psi}_0\rangle$ of the state (29) but numerical evaluations performed with positive values for the parameters s_0 and δ varying in some limited range indicate that the pseudo-norm does not vanish.

Summarizing, we have found that the biorthogonal expansion related to Eq.(31) for $\tilde{H}$ is incomplete with appearance of states $\tilde{\Psi}_{n,n\pm1}$ of zero pseudo-norm. In one-dimensional Quantum Mechanics this is associated to non-diagonalizability. In our two-dimensional case

[e] It is necessary to remind here again that $\tilde{\Phi}_{0,1}$ is only a particular solution of (32).

we have not prooven the existence of associated functions which are normalizable. Irrespectively of that we have discovered an additional state for $\tilde{H}$ constructed from zero modes of Q^- which is pseudo-orthogonal to the other states. This vector definitely should also enter the biorthogonal expansion for $\tilde{H}$.

Continuing the discussion of pseudo-Hermiticity, an imaginary coordinate shift generates this property for $\tilde{H}(\vec{x}+i\vec{\delta})$, since the following equation holds:

$$\tilde{H}(\vec{x}+i\vec{\delta}) = \tilde{H}^{\star}(\vec{x}-i\vec{\delta}) = \exp{(-2i\delta\partial)}\tilde{H}^{\dagger}(\vec{x}+i\vec{\delta})\exp{(+2i\delta\partial)}. \tag{41}$$

Comparing with (2), one can conclude that the explicit form of operator η in (2) can be written as:

$$\eta_{\delta} \equiv \exp{(+2i\vec{\delta}\vec{\partial})} \equiv O^{\dagger}O; \quad O \equiv \exp{(+i\vec{\delta}\vec{\partial})} \equiv O^{\dagger}. \tag{42}$$

In terms of η (from now on the dependence on δ is not written explicitly), the new (pseudo)inner product is defined [8] as:

$$\langle\Omega(\vec{x})|\Gamma(\vec{x})\rangle_{\eta} \equiv \langle\Omega(\vec{x})|\eta\Gamma(\vec{x})\rangle. \tag{43}$$

The precise form (42) of η gives for arbitrary $\Omega(\vec{x})$ and $\Gamma(\vec{x})$:

$$\langle\Omega(\vec{x})|\Gamma(\vec{x})\rangle_{\eta} \equiv \langle\Omega(\vec{x})|\eta\Gamma(\vec{x})\rangle = \langle O\Omega(\vec{x})|O\Gamma(\vec{x})\rangle = \int d^2x\Big(\Omega(\vec{x}+i\vec{\delta})\Big)^{\star}\Gamma(\vec{x}+i\vec{\delta}). \tag{44}$$

It is clear now, why the pseudometric η is positively definite: for $\Omega=\Gamma$ the $\eta-$norm is equal to the integral of $|\Omega(\vec{x}+i\vec{\delta})|^2$.

5 Conclusions.

Higher order (nonlinear) SUSY algebra has allowed us to construct an **exactly solvable** two-dimensional non-Hermitian quantum model. We stress that this model *is not amenable* to separation of variables, and it can be considered as a specific $PT-$non-symmetric complexified version of generalized two-dimensional Morse model with additional $\sinh^{-2}$ term. The spectrum of the model is real. Here we focused attention on the property of pseudo-Hermiticity of the model. To our knowledge this is the first time that pseudo-Hermiticity is

realized explicitly for a nontrivial two-dimensional case. Following the general results, we also studied the biorthogonal expansion and the metric operator associated to pseudo-Hermiticity. In particular, it was shown that the Hamiltonian of the model is not diagonalizable.

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